4201. There are $6^3 = 216$ outcomes in the possibility space. Consider the outcomes in which the first two $X+Y$ add up to the third $Z$, classified by the value of $Z$:

| $Z$ | $(X, Y)$ |
|---|---|
| 1 | None |
| 2 | $(1, 1)$ |
| 3 | $(1, 2), (2, 1)$ |
| ... | ... |

Consider also the possibility that either $X$ or $Y$ could be largest.

4202. Find the equation of the normal at $(p, p^2)$. Solve this simultaneously with $y = x^2$, to show that the other endpoint of the chord is at $x = -p - \frac{1}{2p}$. Find the $x$ component of the length of the chord, and compare it to $2p$.

— Alternative Method —

Consider the symmetry of the scenario.

4203. You could rephrase "The smallest set which can be guaranteed to contain the range of the function" as "The broadest possible range." In each case, work out the greatest and least possible value of the function in question.

4204. (a) Sketch $y = 2^x$ and $y = 3^x$ first, then reflect them in the line $y = x$.
(b) Take $\log_a x$, and raise base and input to the same power, in order to convert the base to $b$.

4205. This is slightly easier by parts, given the algebraic form of the result. Let $u = 3x - 6$.

— Alternative Method —

You can use substitution, with $u = 2x + 3$.

4206. Express $\tan\theta$ in terms of $p$. Then square both sides, reciprocate, add 1, use a Pythagorean trig identity and finally reciprocate back.

4207. Assume, for a contradiction, that $\sqrt[3]{2}$ is rational, so can be written as $p/q$, where $p, q \in \mathbb{N}$. Let $p$ have $m$ factors of 2, and $q$ have $n$ factors of 2.

$$\begin{aligned} \sqrt[3]{2} &= \frac{p}{q} \\ \Longrightarrow 2 &= \frac{p^3}{q^3} \\ \Longrightarrow 2q^3 &= p^3. \end{aligned}$$

Set up a new equation for the number of factors of 2 on each side of the above equation. This will lead to a contradiction.

4208. (a) Write $\frac{dv}{dt} = kv^2$. Then divide both sides by $v^2$ and integrate both sides *with respect to $t$*.
(b) Use the (integral) chain rule, which is the same as the parametric integration formula, to turn the LHS from an integral wrt $t$ to an integral wrt $v$. Then carry out the integrals. You might need to rename a constant as you go.

4209. Solve the boundary equation $\sec x = \operatorname{cosec} x$ for $x \in [0, 2\pi)$. Then sketch the curves $y = \sec x$ and $y = \operatorname{cosec} x$, noting that both have sign changes at vertical asymptotes. Compare the $y$ values. The solution set takes the form $A \cup B \cup C$, in which the sets $A, B, C$ are intervals.

4210. Set up the equation for intersections. It is a quadratic in $x^2$. There are two ways in which this could yield a double root, thus a point of tangency:
① $x^2 = 0$ could be a root of the quadratic.
② The quadratic could have $\Delta = 0$.

4211. Name the central point $X$, and then use a vector method to show that $\overrightarrow{AB} = \overrightarrow{DC}$.

4212. Two of these can be calculated.

4213. Don't use calculus here. A positive octic must have a range of the form $[a, \infty)$. Complete the square on the quadratic $1 + x + x^2$.

4214. Use a small-angle approximation for $\cos\theta$, and the binomial expansion for $\left(1+\theta^2\right)^{-1}$. Neglect terms above $\theta^2$. Multiply the two expansions together, again neglecting the term above $\theta^2$. Once you have a quadratic in $\theta$, carry out the definite integral.

4215. Differentiate implicitly, and set $\frac{dy}{dx} = 0$. You'll get

$$x = \frac{k - y}{2y^2}.$$

Substitute into the original equation, and set up a quadratic in $y$ with coefficients in $k$. Solve this for $y$. Then consider that $k$ is large and +ve.

4216. You are looking for a generalised version of

$$\int \frac{\mathrm{f}'(x)}{(\mathrm{f}(x))^2}\,dx = -\frac{1}{\mathrm{f}(x)} + c.$$

4217. Consider the boundary cases, assuming that
① the block is on the point of sliding up the slope, so $F_{\text{max}}$ acts down the slope,
② the block is on the point of sliding down the slope, so $F_{\text{max}}$ acts up the slope.

4218. Take the index $i$ out of the $j$-indexed sum. Then use the standard result that the sum of the first $n$ integers is $\frac{1}{2}n(n+1)$.

4219. Use log/index rules to get rid of the logarithms from the first equation. Then solve simultaneously with the second equation.

4220. Set up the equation for intersections, in the form $\mathrm{f}(x) = 0$. Show that $\mathrm{f}(x)$ never changes sign.

4221. On an $8 \times 8$ grid, start in the top left, and fill in the number of distinct paths which can be taken to each square. You should soon see the pattern.

——— Alternative Method ———

Find the number of single-square moves required to get to the bottom right, and how many of them must be downwards/rightwards.

4222. Use N-R to find the roots of the polynomial. Then use the definite integration facility on a calculator to find the signed area and so the area.

4223. Rearrange to a cubic. For this to have exactly two real roots, it must have an SP on the $x$ axis.

4224. The iteration is N-R for the equation $\mathrm{f}'(x) = 0$.

4225. (a) Consider cubics which are translations.
(b) Consider $k \in \mathbb{Z}^-$.

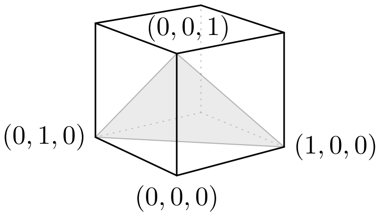
4226. Set up a definite integral between $x = 0$ and $x = k$. Equate this to 18. The resulting equation is a quadratic in $k^{\frac{2}{3}}$.

4227. (a) Just try it!
(b) Multiply top and bottom of the fraction by the factor $(2^{\frac{2}{3}} - 2^{\frac{1}{3}} + 1)$. Multiplying out, all irrational terms should cancel.

4228. (a) Use the chain rule in $\dfrac{d^2y}{dx^2} \equiv \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)$.
(b) Find the first derivative, in simplified terms of $t$, using the usual parametric differentiation formula. Substitute this into the formula from part (a), and look for roots and a sign change.

4229. Since $\overrightarrow{AB}$ is parallel to $\mathbf{i} + 3\mathbf{j}$, we know that $\Delta y$ is three times $\Delta x$. Write this algebraically and solve.

4230. Consider the possibility space as a cube of side length 1. The boundary of the successful region is the triangle shown:




4231. Calculate the horizontal and vertical components of velocity at $P$, together with the horizontal and vertical positions (from an origin at the bottom left of the diagram) at $P$. Set up a vertical *suvat* to find the time of flight. Then set up a horizontal *suvat* to find the horizontal range from $P$. Add this to the horizontal position of $P$.

4232. (a) Substitute the definitions and solve.
(b) Substitute the definitions. Write the resulting expression as a proper algebraic fraction, then in partial fractions. Expand each using the generalised binomial expansion.

4233. (a) For the vertical asymptote, look for the roots of the denominator. For the horizontal, write the fraction in the form
$$\frac{6x+1}{2x-9} \equiv A + \frac{B}{2x-9}.$$
(b) Differentiate by the quotient rule.
(c) The graph is a transformed reciprocal graph. Draw the asymptotes first, and the position of the graph follows.

4234. Rewrite the RHS over base 2, then you can equate the indices and solve.

4235. (a) Solve for intersections, and factorise. Show that one intersection is at a repeated root.
(b) Consider the parity of the root.

4236. Write $\frac{5\pi}{12} = \frac{2\pi}{12} + \frac{3\pi}{12}$ and use compound angles.

4237. (a) The $(x, y)$ plane is (horizontal, horizontal). The vertical $z$ axis is represented by the origin $O$. Since there are no other forces acting in the $x$ direction, the reaction force at the hinge acts only in the $y$ direction.
(b) Hinges ensure that a door can only rotate, not translate. So, to prove equilibrium, you need only consider the moments about the $z$ axis.
(c) i. Take moments in the $(x, y)$ plane around anywhere but the hinges.
ii. The hinges also hold the door **up** (z). This doesn't feature in the $(x, y)$ diagram, being perpendicular to it.

4238. Assume wlog that the rotational symmetry is around the origin. In algebra, $\mathrm{h}(-x) \equiv -\mathrm{h}(x)$. Differentiate this statement twice.

4239. (a) Use the generalised binomial expansion.
(b) Take a factor of $x^{\frac{1}{3}}$ out of $(x+h)^{\frac{1}{3}}$, and then substitute in the result of (a). Cancel a factor of $h$ before taking the limit.

4240. Write $a = y$ and $b = x$. Show that the quartic curve $y = x^4$ lies above the parabola $y = x^2 - 4$ for all $x$.

4241. This is about listing the possibilities in such a way as to guarantee that you have them all. Classify them by horizontal, vertical and the two diagonals.

4242. Sketch $y = \arctan x$ (reflection of the invertible branch of $y = \tan x$ in $y = x$). The boundary cases for $y = kx$ are the tangent line to $y = \arctan x$ at the origin, and the $x$ axis.

4243. Rearrange to $X_1 + X_2 < 1$, and then consider the distribution of the variable $Y = X_1 + X_2$.

4244. The quantity $x^2 + y^2$ represents squared distance from the origin of an $(x, y)$ plane. So, find the equation of the normal to the line $ax + by = c$ passing through the origin. Solve simultaneously with $ax + by = c$ to find the point closest to the origin. Sub into $x^2 + y^2$ and simplify.

4245. This doesn't require calculation of side lengths. Just use the symmetry of the graphs.

4246. Break the position vector apart to get

$$\mathbf{r} = \begin{pmatrix}1\\2\\0\\4\end{pmatrix} + t\begin{pmatrix}2\\0\\1\\0\end{pmatrix} + t^2\begin{pmatrix}0\\-4\\0\\3\end{pmatrix}.$$

Ignore the initial position, which isn't relevant, and write displacement as

$$\mathbf{s} = t\begin{pmatrix}2\\0\\1\\0\end{pmatrix} + t^2\begin{pmatrix}0\\-4\\0\\3\end{pmatrix}.$$

Explain why this produces a parabola.

4247. Use a double-angle formula.

4248. Rearrange to $y = \frac{1}{2}\sqrt{x^2 + 9}$. Find the gradient at $x = 4$, and thus the angle of projection. Use this to find initial horizontal/vertical components of velocity. Set up horizontal and vertical *suvat* (including initial position at the end of the ramp) and eliminate $t$ for the Cartesian equation.

4249. Express ${}^n\mathrm{C}_r + {}^n\mathrm{C}_{r+1}$ in factorials using the given formula. Then simplify by putting the fractions over a common denominator. You want to reach ${}^{n+1}\mathrm{C}_{r+1}$, which you might want to pre-express in factorials to see what you're aiming for.

4250. Use the identity $\tan\theta \equiv \cot(90° - \theta)$. This comes from the fact that, if a line is reflected in the line $y = x$, its gradient is reciprocated.

4251. (a) Refer to the linear approximation.
(b) Differentiate both sides twice with respect to $\theta$. Then substitute in $\theta = 0$.
(c) Differentiate again with respect to $\theta$, and sub in $\theta = 0$.

4252. Put each pair over a common denominator.

4253. In both parts, set up the limit

$$\mathrm{A}(k) = \lim_{p\to 0^+} \int_p^1 x^{-k}\, dx.$$

Carry out the definite integral, and show that the resulting limit converges in part (a), but diverges in part (b).

4254. Use the factor theorem to write

$$\mathrm{g}(x) = (x - \alpha)\,\mathrm{p}(x).$$

Differentiate by the product rule and substitute in $x = \alpha$. This should tell you that $\mathrm{p}(x)$ has a factor of $(x - \alpha)$ and that $\mathrm{g}(x)$ therefore has a factor of $(x - \alpha)^2$. Repeat this procedure.

4255. Use the first Pythagorean identity.

4256. Find the equation of a generic normal at point $(p, p^2)$. Show that this normal re-intersects the curve at $x = -\frac{1}{2p} - p$. Look for the value of this expression closest to zero, using calculus.

4257. The compound-angle formula is

$$\tan(\theta + \phi) \equiv \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}.$$

Setting $\theta = \theta_1$ and $\phi = \theta_2 + \theta_3$, and using the notation given in the question,

$$\tan(\theta_1 + \theta_2 + \theta_3) \equiv \frac{x + \tan(\theta_2 + \theta_3)}{1 - x\tan(\theta_2 + \theta_3)}.$$

Continue from here, using the same compound-angle formula again.

4258. (a) Draw force diagrams, including three different reaction forces. Consider two cases: friction is limiting for both blocks, or only for the left-hand block.
(b) Consider explicitly the cases $\mu = \frac{1}{6}$ and $\mu = \frac{1}{5}$. Show that both of these satisfy the conditions of the question.

4259. (a) Multiply by $\cos x$.
(b) There are two SPs.

(c) There are two vertical asymptotes.

(d) Find the $y$ intercept also. Draw asymptotes before the curve. If you're unsure about how to join the dots, test some $x$ values either side of the vertical asymptotes. This will tell you how the thing fits together.

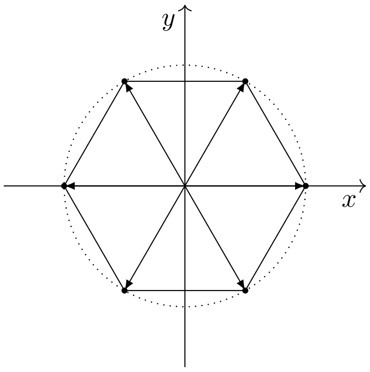
4260. Call the cubic $y = \mathrm{f}(x)$. You know that $\mathrm{f}'(x)$ is a quadratic. Since there are stationary points at $x = p, q$, this quadratic has roots at $x = p, q$. Hence, $\mathrm{f}'(x) = k(ax^2 + bx + c)$, for some constant $k$. Integrate this and then use two facts:

- the curve passes through the origin,
- the curve is monic.

4261. The centres of the circles are at

$$\left(2\cos\frac{k\pi}{3}, 2\sin\frac{k\pi}{3}\right).$$

The position vectors of the centres have length 2, and their directions (anticlockwise from positive $x$) are $\theta = 0, \frac{\pi}{3}, ...$. This puts the centres at the vertices of a regular hexagon of side length 2.




4262. (a) A triple root $x = \alpha$ corresponds to a factor of $(x - \alpha)^3$.

(b) Expand $(x + b)^3$ by the binomial expansion, then multiply by $(x + c)$.

(c) Write down the value of $a$ using the coefficients of $x^4$. Then equate the coefficients of $x^3$ and $x^2$ to produce simultaneous equations in $b$ and $c$. Solve these. You'll get two sets of values. Check them against the coefficient of $x^0$.

4263. Reverse the direction of the second integral, which corresponds to re-evaluating the same area in the opposite direction. This gives

$$\int_0^k \mathrm{f}(x)\,dx + \int_0^k \mathrm{f}(-x)\,dx = 0.$$

Explain why $\mathrm{f}(x) + \mathrm{f}(-x) = 0$ for all $x$.

4264. Solve to find SPs. Then consider the conditions required for your solution to exist: no division by zero, no square roots of negative numbers etc.

4265. (a) Use $\sin^2\theta + \cos^2\theta \equiv 1$.

(b) Differentiate to find $\mathbf{v}$, then use Pythagoras to find $|\mathbf{v}|$.

(c) Rotation combined with constant velocity in a perpendicular direction gives...

4266. (a) Find the $(X, Y)$ of $(1, 0)$. Use Pythagoras to calculate its distance from the origin (of the $(X, Y)$ plane).

(b) Use elimination. Multiply the first equation by $a$ and the second by $b$, then add the equations. This will give $x$. Use a similar procedure to find $y$.

4267. Use a combinatorics approach, in a possibility space of $6^6$ equally likely outcomes. Work out the number of sets of three different scores, e.g. $\{1, 2, 3\}$. Then work out the number of orders of $(1, 1, 2, 2, 3, 3)$.

4268. Label the squares as follows

| | | |
|---|---|---|
| 1 | 2 | 3 |
| 4 | 5 | 6 |
| 7 | 8 | 9 |

Let × start, without loss of generality. Show that, if ○ plays on an even square, i.e. not a corner, then ×, playing logically, will win.

4269. (a) Differentiate the given definition.

(b) Let $y = \mathrm{E}(x)$. Then solve by separation of variables to find the general solution. Use $\mathrm{E}(1) = e$ to produce the value of the constant of integration.

(c) From part (b), you know that

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + ...$$

Substitute $x = 1$. Then split the sum into two parts: the first five terms and all the rest. Then rearrange for the rest, and evaluate.

4270. The RHS is positive. In the original equation, each factor is individually positive. But this is the same as making the whole product positive. So, consider $y = |(x-1)(x-2)(x-3)|$.

4271. Use the vertical asymptote to find $c$. Then, since $y$ tends to the oblique asymptote $y = \frac{1}{2}x - \frac{1}{4}$ for large $x$, the equation of the curve must be expressible, for some constant $k$, as

$$y = \tfrac{1}{2}x - \tfrac{1}{4} + \frac{k}{cx + 4}.$$

Put this over a common denominator, using your known value of $c$, and equate coefficients.

4272. Rearrange and square the first equation to get $\sin^2 x = 2\cos^2 y$. Use the first Pythagorean trig identity to convert sin to cos and vice versa. You'll now have two equations in $\cos x$ and $\sin y$.

Rearrange (put $x$ and $y$ on opposite sides) and square the other equation in a similar manner to the first. Then substitute and solve.

4273. You are given $u_1 = a$ and $u_n = b$. If $a = b = 0$, then the sequence is the zero sequence and all terms are trivially known. So, you can assume that $a, b \neq 0$.

Using the ordinal formula for a GP, $b = ar^{n-1}$, which you can rearrange to $r^{n-1} = \frac{b}{a}$. Consider roots $r = ...$ of this equation in the cases

① $n$ is even,

② $n$ is odd.

4274. Using Pythagoras, set up an expression for $|\mathbf{b}|$. Simplify with the second Pythagorean identity.

— Alternative Method —

Consider the range of the sec function.

4275. Classify by the largest number of black beads $n$ in a single group. With the white beads represented as dots, fill in the following:

| $n$ | 4 | 3 | 2 | 1 |
|---|---|---|---|---|
| | BBBB.... | BBB.B... | ? | ? |
| | | ? | ? | |

Note: there are more arrangements to find than there are question marks in the above.

4276. (a) Use the product rule, factorising the first derivative before differentiating it again.

(b) Find intercepts and behaviour as $x \to \pm\infty$.

4277. (a) Draw a force diagram in limiting equilibrium. Set up three equations: horizontal, vertical and moments around the foot of the ladder.

(b) Repeat the method of part (a). You should get a simple answer in terms of $M$.

4278. Rewrite the second term as a constant plus a proper fraction. This will give you the behaviour as $x \to \pm\infty$. Also, find the vertical asymptote.

4279. The faces are triangles with three edges. So, the ant can't return to $A$ by walking the perimeter of a face. The only paths that return the ant to $A$ circumnavigate the octahedron.

4280. Rearrange the differential equation to

$$\frac{dx}{dt} = \frac{2x}{t} - t^2x^2.$$

Calculate $\frac{dy}{dx}$ for the proposed solution curve, by the quotient rule. Simplify. Then simplify the RHS of the differential equation, and show that you get the same thing.

4281. (a) Firstly, expand and simplify $p^2$ and $q^2$, setting up equations

$$p^2 = ...$$
$$q^2 = ...$$

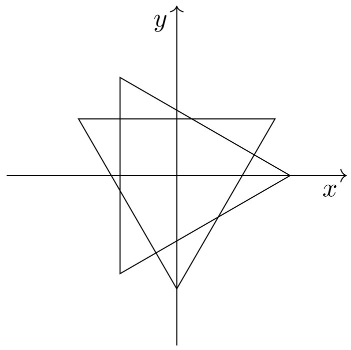
Add and subtract these equations, which will give you the building blocks with which to make $x^2 + xy + y^2$.

(b) The variables $p$ and $q$ run along axes which are the lines $y = x$ and $y = -x$. So, if you prove that the curve is an ellipse in the $(p, q)$ plane, then you have shown that the curve is a rotated ellipse in the $(x, y)$ plane.

4282. Let the upwards acceleration be $a$. Find the height and (upwards) velocity at time $t$. Then take these as the initial condition for freefall back to the ground. Show that the landing speed is $kt$, where $k$ is a constant (depending on acceleration $a$) to be determined.

4283. In each case, the number of $x$ intercepts and the number of vertical asymptotes correspond to the number of roots of the numerator and the number of roots of the denominator.

4284. The scenario, rotating 30° anticlockwise, is




4285. Show that the $f(x) = x^3 - x^2 + x$ is increasing.

4286. The graph is comparable to $y = \sin^2 x$, which can be analysed using a double-angle formula. The only difference is that the $x$ intercepts are now quadruple roots, as opposed to double roots.

4287. Only the mass of liquid can be determined.

4288. This is a quadratic in $t^{\frac{1}{3}}$.

4289. Find the four points of intersection. Then prove these are points of tangency by showing that the former curve is always on or outside the latter.

4290. Use the "angle at the centre" theorem.

4291. You can factorise $\text{h}(x)$ as $\big(|x|+3\big)\big(|x|-2\big)$.

4292. This is a quadratic in $xy$.

4293. (a) It's a two-tailed test.

(b) The critical region should look like

$$\{k \in \mathbb{N} : 0 \leq k \leq a\} \cup \{k \in \mathbb{N} : b \leq k \leq 50\},$$

where $a$ and $b$ are critical values to be found.

(c) Is $x = 11$ in the critical region? In other words, is there sufficient evidence to reject $H_0$?

(d) The phrase "insufficient evidence" is important here. The point being that a test statistic close to, but not in, the critical region does provide evidence against the null hypothesis.

4294. Use the second Pythagorean trig identity.

4295. The graph shown is $(x^2-1)^2y = 1$.

4296. For period 2, you need $x_{n+2} = x_n$. Set this up as an equation in $x$. Simplify the inlaid fractions, to end up with a quadratic in $x$.

4297. The odd and even cases are different.

4298. Since $k_1, k_2 \in \mathbb{N}$, the graph is periodic. It has period $\frac{2\pi}{k}$, where $k = \text{lcm}(k_1, k_2)$.

4299. The normal passes through $\big(p, p^2\big)$ with gradient $-\frac{1}{2p}$. So, it has equation

$$y - p^2 = -\tfrac{1}{2p}(x - p).$$

Solve this simultaneously with $y = x^2$.

4300. Values at which the mod functions switch on are at $y+x-1 = 0$ and $y-x = 0$. These are a pair of perpendicular lines. Anywhere except for on these lines, the locus of $R$ must consist of straight line segments.

——— End of 43rd Hundred ———